

Counting symmetry classes of dissections of a convex regular polygon

Douglas Bowman and Alon Regev
 Department of Mathematical Sciences
 Northern Illinois Univeristy
 DeKalb, IL

Abstract

This paper proves explicit formulas for the number of dissections of a convex regular polygon modulo the action of the cyclic and dihedral groups. The formulas are obtained by making use of the Cauchy-Frobenius Lemma as well as bijections between rotationally symmetric dissections and simpler classes of dissections. A number of special cases of these formulas are studied. Consequently, some known enumerations are recovered and several new ones are provided.

1. Introduction

In 1963 Moon and Moser [13] enumerated the equivalence classes of triangulations of a regular convex n -gon modulo the action of the dihedral group D_{2n} . A year later, Brown [2] enumerated the equivalence classes of these triangulations modulo the action of the cyclic group Z_n . Recall that the triangulations of an n -gon are in bijection with the vertices of the associahedron of dimension $n - 3$ (see Figure 1). Lee [11] showed that the associahedron can be realized as a polytope in $(n - 3)$ -dimensional space having the dihedral symmetry group D_{2n} . Thus Moon and Moser's result and Brown's result are equivalent to enumerating of the vertices of the associahedron modulo the dihedral action and the cyclic action, respectively. The enumeration by Moon and Moser also arose recently in the work of Ceballos, Santos and Ziegler [7]. Their work describes a family of realizations of the associahedron (due to Santos), and proves that the number of normally non-isomorphic realizations is the number of triangulations of a regular polygon modulo the dihedral action. In this paper we generalize the results of Moon and Moser, as well as Brown, and enumerate the number of *dissections* of regular polygons modulo the dihedral and cyclic actions.

Definition 1. Let $n \geq 3$. A k -*dissection* of an n -gon is a partition of the n -gon into $k + 1$ polygons by k non-crossing diagonals. A *triangulation* is an $(n - 3)$ -dissection of an n -gon and an *almost-triangulation* is an $(n - 4)$ -dissection. Let $G(n, k)$ be the set of k -dissections of an n -gon, and let $G(n) = \bigcup_{k=0}^{n-3} G(n, k)$.

In terms of associahedra, a k -dissection corresponds to an $(n - k - 3)$ -dimensional face on an associahedron of dimension n . A natural generalization of the results of Moon and Moser and of Brown is the enumeration of $G(n, k)/D_{2n}$ and $G(n, k)/Z_n$, the sets of cyclic

and dihedral classes, respectively, in $G(n, k)$. In 1978, Read [15] considered an equivalent problem. He enumerated certain classes of cellular structures, which are in bijection with $G(n, k)/D_{2n}$ and $G(n, k)/Z_n$. Read found generating functions for the number of such classes, and included tables of values [15, Tables 3 and 5]. In fact, the first diagonal of Table 5 of Read corresponds to the sequence found by Moon and Moser, and the first diagonal of Table 3 of Read corresponds to the sequence found by Brown. Lisonek [12] studied these results of Read and showed that the sequences $|G(n, k)/D_{2n}|$ and $|G(n, k)/Z_n|$ are “quasi-polynomial” in n when k is fixed. (Here and throughout, $|X|$ denotes the cardinality of a finite set X .) More recently, Read and Devadoss [8] studied various equivalence relations on the set of polygonal dissections. They gave a sequence of figures [8, Figures 22-25] representing all the dihedral classes of n -gons for $3 \leq n \leq 9$. However, none of the above authors give an explicit formula for $|G(n, k)/D_{2n}|$ or for $|G(n, k)/Z_n|$. The present authors [3] give an explicit formula enumerating $G(n, n-4)/D_{2n}$, the dihedral classes of almost-triangulations, equivalently, of edges of associahedra. This formula agrees with the values of the second diagonal of Table 5 of Read [15].

Explicit formulas for $|G(n, k)/Z_n|$ and $|G(n, k)/D_{2n}|$ could in principle be derived from Read’s iteratively defined generating functions, but the resulting formulas would be considerably more complicated than those computed here; see equations (6) and (7). Our approach to solving these enumeration problems is similar to that of Moon and Moser [13]. For each element of the dihedral group, the number of dissections in $G(n, k)$ which are fixed under its action is computed. The Cauchy-Frobenius Lemma is then used to derive the number of dihedral and cyclic classes in $G(n, k)$.

In Section 5 we introduce a combinatorial bijection (26) between certain rotationally symmetric dissections (*centrally unbordered* dissections, see Definition 16) and a set $G^*(n, k)$ of *marked dissections*, which are dissections with one of their parts distinguished. These marked dissections are easy to generate and enumerate. A bijection for *centrally bordered* dissections is implicit in the proof of Lemma 17. Przytycki and Sikora [14] studied a set of marked dissections $P_i(s, n)$, which is a subset of $G^*(n, k)$; however, the classes of dissections enumerated in [14] are different from those studied here.

Besides their intrinsic interest, bijections involving polygonal dissections have connections to other mathematical structures; for example, Torkildsen [17] proved a bijection between $G(n, n-3)/Z_n$ and the mutation class of quivers of Dynkin type A_n , while Przytycki and Sikora describe a relationship between their bijection (between $P_i(s, n)$ and another combinatorial structure) and their work in knot theory as well as Jones’ work on planar algebras.

After proving the general formulas (6) and (7) in Sections 2 through 5, special cases are studied in Section 6. Consequently we not only recover known enumerations but are able to provide several that are new. We note several of the interesting special cases here. For example, setting $k = n-3$ in (6) recovers the result of Moon and Moser [13], and setting $k = n-3$ in (7) recovers the result of Brown [2]. Setting $k = n-4$ in (6) recovers the result of the authors [3], while the following theorem gives a formula for the number of cyclic classes in the case $k = n-4$. For a nonnegative integer n , let C_n denote the n -th Catalan number;

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

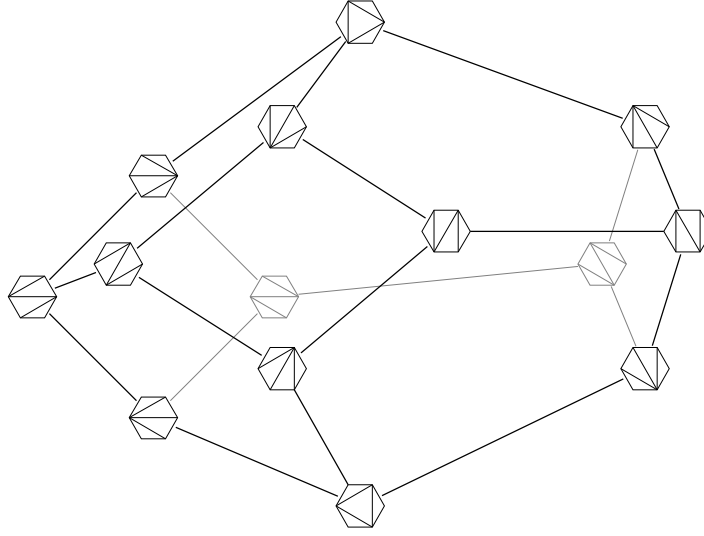


Figure 1: The three-dimensional associahedron

and let $C_n = 0$ otherwise.

Theorem 2. *Let $n \geq 4$. The number $|G(n, n-4)/Z_n|$ of almost-triangulations of an n -gon (equivalently, edges of the $(n-3)$ -dimensional associahedron) modulo the cyclic action is given by*

$$\frac{n-3}{2n}C_{n-2} + \frac{1}{2}C_{n/4-1} + \frac{1}{4}C_{n/2-1}. \quad (1)$$

Setting $k = n - 5$ gives the following formulas.

Theorem 3. *Let $n \geq 5$.*

1. *The number $|G(n, n-5)/Z_n|$ of $(n-5)$ -dissections of an n -gon (equivalently, the number of two-dimensional faces of the $(n-3)$ -dimensional associahedron) modulo the cyclic action is given by*

$$\frac{(n-3)^2(n-4)}{4n(2n-5)}C_{n-2} + \frac{n-4}{8}C_{n/2-1} + \frac{4}{5}C_{n/5-1}. \quad (2)$$

2. *The number $|G(n, n-5)/D_{2n}|$ of $(n-5)$ -dissections of an n -gon (equivalently, the number of two-dimensional faces of the $(n-3)$ -dimensional associahedron) modulo the dihedral action is given by*

$$\frac{(n-3)^2(n-4)}{8n(2n-5)}C_{n-2} + \frac{2}{5}C_{n/5-1} + \frac{3(n-4)(n-1)}{16(n-3)}C_{n/2-1},$$

if n is even, and

$$\frac{(n-3)^2(n-4)}{8n(2n-5)}C_{n-2} + \frac{2}{5}C_{n/5-1} + \frac{n^2-2n-11}{8(n-4)}C_{(n-3)/2}, \quad (3)$$

if n is odd.

A formula equivalent to (1) occurs in the Online Encyclopedia of Integer Sequences [16, sequence A0003444], while the sequences of (2) and (3) occur there without a formula [16, sequences A0003450 and A0003445].

Finally, setting $k = n - 6$ in (7) gives the following formula.

Theorem 4. *Let $n \geq 6$. The number $|G(n, n - 6)/Z_n|$ of $(n - 6)$ -dissections of an n -gon (equivalently, the number of three-dimensional faces of the $(n - 3)$ -dimensional associahedron) modulo the cyclic action is given by*

$$\frac{(n - 3)(n - 4)^2(n - 5)}{24n(2n - 5)}C_{n-2} + \frac{(n - 4)^2}{4n}C_{n/2-2} + \frac{n - 3}{9}C_{n/3-1} + \frac{1}{3}C_{n/6-1}.$$

Another set of results is obtained by specializing equations (6) and (7) to fixed values of k . Setting $k = 1$ gives the formulas $|G(n, 1)/Z_n| = |G(n, 1)/D_{2n}| = n/2 - 1$ if n is even and $|G(n, k)/Z_n| = |G(n, 1)/D_{2n}| = (n - 3)/2$ if n is odd: these formulas are easy to see directly. In the case $k = 2$, (6) and (7) are more interesting.

Theorem 5. *Let $n \geq 2$.*

1. *The number of 2-dissections of an n -gon modulo the cyclic action is*

$$|G(n, 2)/Z_n| = \begin{cases} \frac{1}{12}n(n - 2)(n - 4), & \text{if } n \text{ is even} \\ \frac{1}{12}(n + 1)(n - 3)(n - 4), & \text{if } n \text{ is odd.} \end{cases}$$

2. *The number of 2-dissections of an n -gon modulo the dihedral action is*

$$|G(n, 2)/D_{2n}| = \begin{cases} \frac{1}{24}(n - 4)(n - 2)(n + 3), & \text{if } n \text{ is even} \\ \frac{1}{24}(n - 3)(n^2 - 13), & \text{if } n \text{ is odd.} \end{cases}$$

Note that Theorem 5 agrees with the result of Lisonek [12] in that the formulas obtained are quasi-polynomials.

1.1. The general formulas

Let $A_k^n = |G(n, k)|$ be the number of k -dissections of an n -gon. Cayley [6] showed that for integers $0 \leq k \leq n - 3$,

$$A_k^n = \frac{1}{k + 1} \binom{n + k - 1}{k} \binom{n - 3}{k}. \quad (4)$$

We take $A_0^2 = 1$ corresponding to the trivial dissection of a digon (2-gon). Otherwise, unless n and k are integers with $0 \leq k \leq n - 3$, let $A_k^n = 0$. Note that

$$A_{n-3}^n = \begin{cases} 0, & \text{if } n = 2 \\ C_{n-2}, & \text{otherwise.} \end{cases} \quad (5)$$

Let $\varphi(n)$ denote Euler's totient function (the number of positive integers less than n that are relatively prime to n). The following two theorems are the main results of this paper.

Theorem 6. Let $1 \leq k \leq n - 3$. Let $|G(n, k)/D_{2n}|$ be the number of k -dissections of an n -gon (equivalently, $(n - k - 3)$ -dimensional faces of the $(n - 3)$ -dimensional associahedron) modulo the dihedral action. If n is even then $|G(n, k)/D_{2n}|$ is given by

$$\begin{aligned}
& \frac{1}{2n} A_k^n + \frac{1}{2} A_{(k-1)/2}^{n/2+1} + \frac{1}{4} A_{k/2}^{n/2+1} \\
& + \sum_{3 \leq d|n} \frac{\varphi(d)}{2d} A_{k/d-1}^{n/d+1} + \sum_{2 \leq d \leq n/3} \frac{\varphi(d)(n+k-d)}{2dn} A_{k/d-1}^{n/d} \\
& + \sum_{\substack{2 \leq d|n; \ r \geq 3; \ n_1 + \dots + n_r = n/d \\ k_1 + \dots + k_r + |\{i: n_i \geq 2\}| = k/d}} \frac{\varphi(d)}{2r} \prod_{i=1}^r A_{k_i}^{n_i+1} \\
& + \frac{1}{4} \sum_{\substack{1 \leq t \leq k \\ n_0 + \dots + n_t = n/2 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}) \\
& + \frac{1}{4} \sum_{\substack{0 \leq t \leq k \\ n_0 + \dots + n_t = n/2-1 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}),
\end{aligned}$$

and if n is odd then $|G(n, k)/D_{2n}|$ is given by

$$\begin{aligned}
& \frac{1}{2n} A_k^n + \sum_{3 \leq d|n} \frac{\varphi(d)}{2d} A_{k/d-1}^{n/d+1} + \sum_{2 \leq d \leq n/3} \frac{\varphi(d)(n+k-d)}{2dn} A_{k/d-1}^{n/d} \\
& + \sum_{\substack{2 \leq d|n; \ r \geq 3; \ n_1 + \dots + n_r = n/d \\ k_1 + \dots + k_r + |\{i: n_i \geq 2\}| = k/d}} \frac{\varphi(d)}{2r} \prod_{i=1}^r A_{k_i}^{n_i+1} \\
& + \frac{1}{2} \sum_{\substack{1 \leq t \leq k \\ n_0 + \dots + n_t = n/2 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}).
\end{aligned}$$

(6)

Theorem 7. Let $n \geq 3$. The number $|G(n, k)/Z_n|$ of k -dissections of an n -gon (equivalently, $(n - k - 3)$ -dimensional faces of the $(n - 3)$ -dimensional associahedron) modulo the cyclic action is given by

$$\begin{aligned}
& \frac{1}{n} A_k^n + \sum_{3 \leq d|n} \frac{\varphi(d)}{d} A_{k/d-1}^{n/d+1} + \sum_{2 \leq d \leq n/3} \frac{\varphi(d)(n+k-d)}{dn} A_{k/d-1}^{n/d} \\
& + \sum_{\substack{2 \leq d|n; \ r \geq 3; \ n_1 + \dots + n_r = n/d \\ k_1 + \dots + k_r + |\{i: n_i \geq 2\}| = k/d}} \frac{\varphi(d)}{r} \prod_{i=1}^r A_{k_i}^{n_i+1}.
\end{aligned}$$

(7)

2. Preliminaries

For any labeled graph H , let $V(H)$ be its set of vertices and let $E(H)$ be its set of edges, defined to be two-element subsets of $V(H)$. We frequently denote the edge $\{x, y\}$ by xy . To a dissection $\Phi \in G(n)$ we associate a labeled graph $(V(\Phi), E(\Phi))$, where $V(\Phi) = \{0, \dots, n-1\}$. To the sides of the n -gon we associate the edges $S(\Phi) = \{01, 12, 23, \dots, (n-2)(n-1), (n-1)0\}$ and to the diagonals of the dissection we associate the rest of the edges of the graph. The *distance* between any two vertices $x, y \in V(\Phi)$ is defined to be the graph-theoretic distance between them in the subgraph $(V(\Phi), S(\Phi))$.

It is easily seen that two crossing diagonals of a convex n -gon correspond to edges ab and cd (with $a < b$ and $c < d$) if and only if

$$a < c < b < d \text{ or } c < a < d < b. \quad (8)$$

Thus if $\Phi \in G(n)$, there are no edges $ab, cd \in E(\Phi)$ satisfying (8). Hereafter we identify dissections with their labeled graphs.

The elements of the dihedral group D_{2n} are denoted using ε , ρ and τ to represent the identity, rotation by $2\pi/n$ and reflection about a symmetry axis (which by convention passes through one of the vertices of the n -gon), respectively. The cyclic group Z_n can be identified with the subgroup of D_{2n} generated by ρ .

We use the notation $[x]_n$ to denote the remainder when x is divided by n . The elements of D_{2n} can be represented by their action on the vertices, $\rho(v) = [v+1]_n$ and $\tau(v) = [-v]_n$.

For any $\sigma \in D_{2n}$, let $G(n, k; \sigma)$ denote the subset of $G(n, k)$ consisting of dissections fixed under the action of σ , and let

$$G(n; \sigma) = \bigcup_{k=0}^{n-3} G(n, k; \sigma).$$

Thus if $\Phi \in G(n)$ then $\Phi \in G(n; \rho^i)$ if and only if

$$[x+i]_n[y+i]_n \in E(\Phi) \text{ whenever } [x]_n[y]_n \in E(\Phi),$$

and $\Phi \in G(n; \tau\rho^i)$ if and only if

$$[i-x]_n[i-y]_n \in E(\Phi) \text{ whenever } [x]_n[y]_n \in E(\Phi).$$

The Cauchy-Frobenius Lemma [4] gives the equations

$$|G(n, k)/D_{2n}| = \frac{1}{2n} \left(\sum_{i=0}^{n-1} |G(n, k; \rho^i)| + \sum_{i=0}^{n-1} |G(n, k; \tau\rho^i)| \right) \quad (9)$$

and

$$|G(n, k)/Z_n| = \frac{1}{n} \sum_{i=0}^{n-1} |G(n, k; \rho^i)|. \quad (10)$$

Equations (9) and (10) reduce the problem of enumerating the dihedral and cyclic classes in $G(n, k)$ to that of finding $G(n, k; \sigma)$ for each $\sigma \in D_{2n}$. In fact, the following lemma shows that it suffices to consider only a subset of D_{2n} .

Lemma 8. *Let $0 \leq i \leq n - 1$. Then*

1.

$$|G(n, k; \rho^i)| = |G(n, k; \rho^{\gcd(n, i)})|.$$

2. (a) *If n and i are even then $|G(n, k; \tau \rho^i)| = |G(n, k; \tau)|$.*
 (b) *If n is even and i odd then $|G(n, k; \tau \rho^i)| = |G(n, k; \tau \rho)|$.*
 (c) *If n is odd then $|G(n, k; \tau \rho^i)| = |G(n, k; \tau)|$.*

Proof. 1. Since ρ^i and $\rho^{\gcd(n, i)}$ generate the same subgroup of D_{2n} , they fix precisely the same elements of $G(n, k)$.

2. It is well known [1, p. 243] that conjugate elements in a group acting on a set have the same number of fixed points. The results then follow from the conjugacy relations in D_{2n} .

□

Thus equations (9) and (10) imply

$$|G(n, k)/D_{2n}| = \begin{cases} \frac{1}{2n} \sum_{d|n} \varphi(d) |G(n, k; \rho^{n/d})| + \frac{1}{4} |G(n, k; \tau)| + \frac{1}{4} |G(n, k; \tau \rho)|, & \text{if } n \text{ is even} \\ \frac{1}{2n} \sum_{d|n} \varphi(d) |G(n, k; \rho^{n/d})| + \frac{1}{2} |G(n, k; \tau)|, & \text{if } n \text{ is odd} \end{cases} \quad (11)$$

and

$$|G(n, k)/Z_n| = \frac{1}{n} \sum_{d|n} \varphi(d) |G(n, k; \rho^{n/d})|. \quad (12)$$

Theorems 6 and 7 follow from calculating the terms in (11) and (12), respectively. Section 3 addresses the terms $|G(n, k; \tau)|$ and $|G(n, k; \tau \rho)|$, and Section 5 addresses the terms $|G(n, k; \rho^{n/d})|$.

3. Axially symmetric dissections

The sets $G(n, k; \tau)$ and $G(n, k; \tau \rho)$ of axially symmetric dissections can be enumerated by considering the number of *perpendiculars*, i.e., diagonals of a dissection which are perpendicular to the axis of symmetry; these diagonals have the form $[v]_n[-v]_n$. Denote by $G(n, k; \tau; t)$ the set of dissections in $G(n, k; \tau)$ with exactly t perpendiculars. The notation $G(n, k; \tau \rho; t)$ is defined analogously. Thus

$$G(n, k; \tau) = \sum_{t \geq 0} G(n, k; \tau; t),$$

with the analogous formula holding for $G(n, k; \tau \rho; t)$.

Lemma 9. *If n is even then*

$$|G(n, k; \tau; 0)| = A_{(k-1)/2}^{n/2+1} + A_{k/2}^{n/2+1} \quad (13)$$

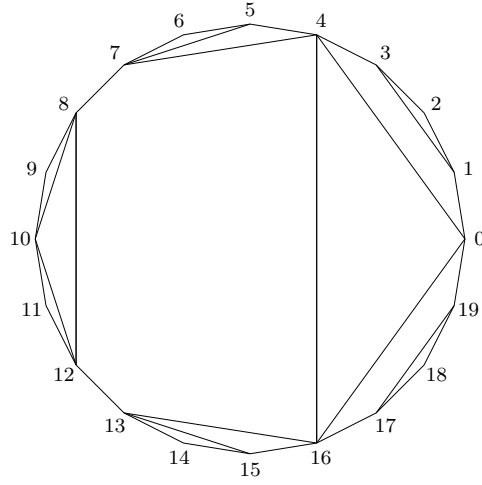


Figure 2: An axially symmetric dissection.

and for $t \geq 1$,

$$|G(n, k; \tau; t)| = \sum_{\substack{n_0 + \dots + n_t = n/2 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}). \quad (14)$$

Proof. Let $[v_1]_n[-v_1]_n, \dots, [v_t]_n[-v_t]_n \in E(\Phi)$ be the perpendiculars of Φ , where $t \geq 0$ and $0 = v_0 < v_1 < \dots < v_t < v_{t+1} = n/2$. Let $n_s = v_{s+1} - v_s$ for $s = 0, \dots, t$. The case $t = 0$ is considered separately since in this case $v_0 v_1 = 0 \frac{n}{2} = [-0]_n[-\frac{n}{2}]_n = [-v_0]_n[-v_1]_n$, while in all other cases $v_s v_{s+1} \neq [-v_s]_n[-v_{s+1}]_n$. Let $\Phi \in G(n, k; \tau; 0)$ and suppose first that $0 \frac{n}{2} \in E(\Phi)$. The remaining $k-1$ diagonals of Φ are equally distributed between the two sides of the symmetry axis. Each such dissection then uniquely corresponds to a dissection of the resulting $(n/2 + 1)$ -gon on either of its sides. Thus there are $A_{(k-1)/2}^{n/2+1}$ dissections of this type.

By a similar argument, there are $A_{k/2}^{n/2+1}$ dissections in $G(n, k; \tau; 0)$ which do not contain the diagonal $0 \frac{n}{2}$, and (13) follows. Now suppose $1 \leq t \leq k$ and let $\Phi \in G(n, k; \tau; t)$. The $k-t$ other diagonals of Φ are pairs of the form xy and $[-x]_n[-y]_n$, where $v_s \leq x < y \leq v_{s+1}$ and $0 \leq s \leq t$. Each such diagonal xy is either of the form $v_s v_{s+1}$ or it is a diagonal of the $(n_s + 1)$ -gon with vertices $v_s, v_s + 1, \dots, v_{s+1}$. Let k_s be half the number of diagonals in the region defined by the vertices $v_s, v_{s+1}, [-v_{s+1}]_n$ and $[-v_s]_n$. If $v_s v_{s+1} \in E(\Phi)$ then the dissection of this region corresponds to a $(k_s - 1)$ -dissection of the $(n_s + 1)$ -gon. Otherwise, it corresponds to a k_s -dissection of the $(n_s + 1)$ -gon. This proves (14). \square

Figure 2 gives an example of an axially symmetric dissection where, using the notation above, $n = 20$, $k = 12$, $t = 2$, $v_1 = 4$, $v_2 = 8$, $k_0 = 2$, $k_1 = 2$ and $k_2 = 1$. The dissection of the region $s = 0$ corresponds to a 1-dissection of the pentagon with vertices 0, 1, 2, 3, 4, and the dissection of the region $s = 1$ corresponds to a 2-dissection of the pentagon with vertices 4, 5, 6, 7, 8.

For the next two lemmas, $v_s v_{s+1} \neq [-v_s]_n[-v_{s+1}]_n$ for all s , and therefore the case $t = 0$ need not be considered separately. The proofs are otherwise analogous to that of Lemma 9.

Lemma 10. *If n is even then for any $t \geq 0$*

$$|G(n, k; \tau\rho; t)| = \sum_{\substack{n_0 + \dots + n_t = n/2 - 1 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}). \quad (15)$$

Lemma 11. *If n is odd then for $t \geq 0$,*

$$|G(n, k; \tau; t)| = \sum_{\substack{n_0 + \dots + n_t = (n-1)/2 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}). \quad (16)$$

4. Components and marked dissections

A dissection $\Phi \in G(n)$ can be associated with the set $\mathcal{C}(\Phi)$ of *components* comprising it, each of these components being a polygon free of dissecting diagonals. Thus a component is a subgraph γ of Φ such that for some $r \geq 2$ and $0 \leq v_r = v_0 < \dots < v_{r-1} \leq n-1$,

$$\begin{aligned} V(\gamma) &= \{v_0, \dots, v_{r-1}\}, \\ E(\gamma) &= \{v_0v_1, \dots, v_{r-1}v_r\}, \end{aligned}$$

and

$$v_i v_j \in E(\Phi) \implies v_i v_j \in E(\gamma). \quad (17)$$

For example, if Φ is the dissection shown in Figure 2 then $\mathcal{C}(\Phi)$ consists of 32 digons, 10 triangles, two quadrilaterals and one hexagon. For $r \geq 2$ let $\mathcal{C}_r(\Phi)$ be the set of r -gons in $\mathcal{C}(\Phi)$, which we call *r -components*.

In what follows, X represents any list of parameters. For any $r \geq 2$, an *r -marked dissection* is a dissection Φ with one of its r -components γ distinguished. Let

$$G^r(X) = \{(\Phi, \gamma) : \Phi \in G(X), \gamma \in \mathcal{C}_r(\Phi)\}$$

be the set of r -marked dissections associated with $G(X)$, and let $G^*(X) = \bigcup_{r \geq 2} G^r(X)$.

Lemma 12. *Let $0 \leq k \leq n-3$. Then*

$$|G^2(n, k)| = (n+k)A_k^n, \quad (18)$$

and for $r \geq 3$,

$$r|G^r(n, k)| = n \sum_{\substack{n_1 + \dots + n_r = n \\ k_1 + \dots + k_r + |\{i: n_i \geq 2\}| = k}} \prod_{i=1}^r A_{k_i}^{n_i+1}. \quad (19)$$

Proof. Equation (18) holds since $|\mathcal{C}_2(\Phi)| = n + k$ for any dissection $\Phi \in G(n, k)$. Let $r \geq 3$. The left-hand side of (19) enumerates the marked dissections having one of the r vertices v_0 in the distinguished r -gon distinguished. The right-hand side enumerates the same elements by selecting the r -component first and then dissecting the region between each side of the component and the n -gon. Choose $0 \leq v_0 \leq n - 1$. Decompose the cycle $v_0[v_0+1]_n, [v_0+1]_n[v_0+2]_n, \dots, [v_0+n-1]_n[v_0+n]_n$ into consecutive paths of length n_i (where $1 \leq i \leq r$ and $n_i \geq 1$) with vertices v_0, v_1, \dots, v_r . Observe that every such decomposition of the edges of the n -gon corresponds to a set of n_i satisfying $n_1 + \dots + n_r = n$. The vertices $V(\gamma) = \{v_0, \dots, v_{r-1}\}$ of an r -component are thus determined by

$$v_i = [v_0 + \sum_{j=1}^i n_j]_n.$$

Now for each $0 \leq i \leq r - 1$ select a dissection of the region between the edge $v_i v_{i+1}$ and the path from v_i to v_{i+1} along the sides of the n -gon. Each such dissection corresponds to a set of k_i satisfying $k_1 + \dots + k_r + |\{i : n_i \geq 2\}| = k$. (The term $|\{i : n_i \geq 2\}|$ accounts for those sides of the r -component that are not sides of the n -gon). Finally for each k_i there $A_{k_i}^{n_i+1}$ such dissections. \square

Since a triangulation consists of $n - 2$ triangles, a simpler formula for 3-marked triangulations is

$$|G^3(n, n - 3)| = (n - 2)A_{n-3}^n. \quad (20)$$

Definition 13. Let $\Phi \in G(n)$ and consider the representation of Φ as a set of points in the interior or boundary of a regular n -gon embedded in \mathbb{R}^2 and centered at the origin. Any component of Φ is a subset of the planar region. There is thus a unique component of Φ containing the origin. We call this component the *central polygon* $Z(\Phi)$ of a dissection Φ . Let $G_m(X)$ be the subset of $G(X)$ consisting of dissections whose central polygon is an m -gon;

$$G_m(X) = \{\Phi \in G(X) : Z(\Phi) \in \mathcal{C}_m(\Phi)\},$$

and put $G_{\neq m}(X) = G(X) \setminus G_m(X)$.

For $\Phi \in G_m(n, k)$, if the regions outside of $Z(\Phi)$ are triangulated then $m = n - k$. More generally,

$$m \leq n - k \text{ for any } \Phi \in G_m(n, k). \quad (21)$$

Definition 14. Let $\Phi \in G(n)$. Given an edge $xy \in E(\Phi)$ and a vertex $v \in V(\Phi)$, we say that v is *outer to* xy if v lies strictly between x and y on the shorter path of the n -gon connecting them.

Remark 15. A vertex $v \in V(Z(\Phi))$ cannot be outer to any edge $xy \in E(\Phi)$.

5. Rotationally symmetric dissections

The enumeration of the sets $G(n, k; \rho^{n/d})$ of rotationally symmetric dissections can be achieved by considering separately the following two classes.

Definition 16. A dissection $\Phi \in G(n, k; \rho^{n/d})$ is said to be *centrally bordered* if $\Phi \in G_d(n, k; \rho^{n/d})$ and *centrally unbordered* if $\Phi \in G_{\neq d}(n, k; \rho^{n/d})$.

Lemma 17 addresses the case of centrally bordered dissections; Lemmas 22, 24 and 25 and Theorem 26 address the case of centrally unbordered dissections. When considering the set $G(n; \rho^{n/d})$ it is convenient to put $j = n/d$. Let δ_{xy} denote the Kronecker delta.

Lemma 17. Let $d, j \geq 2$ and let $0 \leq k \leq n - 3$.

$$|G_d(n, k; \rho^j)| = j A_{(k-d+\delta_{d2})/d}^{j+1}. \quad (22)$$

Proof. Let $\Phi \in G_d(n, k; \rho^j)$. By symmetry the central polygon $Z(\Phi)$ is a regular d -gon, which can be positioned in j different ways in the n -gon. Since all $d - \delta_{d2}$ edges of $Z(\Phi)$ are diagonals of Φ , there are $k - d + \delta_{d2}$ diagonals of Φ which are not edges of $Z(\Phi)$. By symmetry these diagonals are equally distributed among the d resulting $(j + 1)$ -gons, giving $A_{(k-d+\delta_{d2})/d}^{j+1}$ choices for each position of $Z(\Phi)$. \square

Enumeration of the centrally unbordered dissections $G_{\neq d}(n, k; \rho^{n/d})$ is accomplished by the introduction of a bijection with the marked dissections which were enumerated in Lemma 12. We define the following “furling” maps F_d and F_d^* (see Figure 3).

Definition 18. Let $d, j \geq 2$.

1. Define a function f_d on edges of a graph of n vertices by $f_d(xy) = [x]_j[y]_j$.
2. Let Φ be a dissection or a component of a dissection of an n -gon. Define the labeled graph $F_d(\Phi)$ by $V(F_d(\Phi)) = \{[x]_j : x \in V(\Phi)\}$ and $E(F_d(\Phi)) = f_d[E(\Phi)]$.¹
3. For $\Phi \in G_{\neq d}(n, k; \rho^j)$, define $F_d^*(\Phi) = (F_d(\Phi), F_d(Z(\Phi)))$.

The conclusions of the following remark are easy observations.

Remark 19. Let $d, j \geq 2$.

- (i) If $\Phi \in G_{\neq d}(n, k; \rho^j)$ then its central polygon $Z(\Phi)$ is itself invariant under ρ^j and hence $Z(\Phi)$ is an rd -gon for some $r \geq 2$. Therefore $G_{\neq d}(n, k; \rho^j)$ can be partitioned as follows.

$$G_{\neq d}(n, k; \rho^j) = \bigcup_{r \geq 2} G_{rd}(n, k; \rho^j). \quad (23)$$

- (ii) If $\Phi \in G_{\neq d}(n, k; \rho^j)$ and $xy \in E(\Phi)$, then the distance between x and y is at most $j - 1$.
- (iii) Suppose $\Phi \in G_{\neq d}(n, k; \rho^j)$. From the previous observation and by symmetry, it follows that for $0 \leq x < y \leq j - 1$,

$$xy \in E(F_d(\Phi)) \text{ if and only if either } xy \in E(\Phi) \text{ or } y(x + j) \in E(\Phi). \quad (24)$$

¹Functions on subsets of a set are defined in the usual way and denoted using square brackets.

The map \mathbf{u}_d will be used to output symmetrically distributed edges in a dissection.

Definition 20. Let $d, j \geq 2$. For an edge ab of a dissection $\Phi \in G(j)$, define

$$\mathbf{u}_d(ab) = \bigcup_{0 \leq i \leq d-1} \left\{ \{[a + ij]_n, [b + ij]_n\} \right\}.$$

Note that if $\Phi \in G_{rd}(n, k; \rho^j)$ and $\gamma = Z(\Phi)$ then by the d -fold symmetry, for some $0 \leq v_r = v_0 < \dots < v_{r-1} \leq j - 1$,

$$V(\gamma) = \{v_0, \dots, v_{r-1}, v_0 + j, \dots, v_{r-1} + j, \dots, v_{r-1} + (d-1)j\},$$

$$E(\gamma) = \mathbf{u}_d(v_0v_1) \cup \mathbf{u}_d(v_1v_2) \cup \dots \cup \mathbf{u}_d(v_{r-2}v_{r-1}) \cup \mathbf{u}_d(v_{r-1}(v_0 + j)) \quad (25)$$

The following lemma is readily verified using (24) and Remark 15. It employs the notation (25) for $Z(\Phi)$.

Lemma 21. Let $d, j \geq 2$ and let $\Phi \in G_{rd}(n, k; \rho^j)$. Consider the map $f_d : E(\Phi) \rightarrow E(F_d(\Phi))$ defined above. For an edge $ab \in E(F_d(\Phi))$, with $a < b$, the preimage of ab under f_d is given by:

$$f_d^{-1}[ab] = \begin{cases} \mathbf{u}_d(ab), & \text{if } a > v_0 \text{ or } b < v_{r-1}, \\ \mathbf{u}_d(b(a + j)), & \text{if } a \leq v_0, b \geq v_{r-1}, \text{ and } ab \neq v_0v_1, \\ \mathbf{u}_d(ab) \cup \mathbf{u}_d(b(a + j)), & \text{if } ab = v_0v_1. \end{cases} \quad (26)$$

Lemma 22. Let $d, j, r \geq 2$ and let $1 \leq k \leq n - 3$. If $\Phi \in G_{rd}(n, k; \rho^j)$ then

$$F_d^*(\Phi) \in G^r(j, k/d - \delta_{r2}).$$

Proof. Clearly $F_d(\Phi)$ has j vertices. We show that its diagonals are noncrossing. Suppose that a_1b_1 and a_2b_2 are crossing diagonals of $F_d(\Phi)$, with $0 \leq a_1 < a_2 < b_1 < b_2 \leq j - 1$. By Lemma 21, for each $i = 1, 2$ either a_ib_i or $a_i(b_i + j)$ is a diagonal of Φ . Since $a_1 < a_2 < b_1 < b_2 < a_1 + j < a_2 + j$, these two diagonals of Φ are crossing. This contradiction shows that $F_d(\Phi) \in G(j)$.

We next show that $F_d(Z(\Phi))$ is an r -component of $F_d(\Phi)$. By symmetry the center $Z(\Phi)$ has the form (25). Therefore $V(F_d(Z(\Phi))) = \{v_0, \dots, v_{r-1}\}$ and $E(F_d(Z(\Phi))) = \{v_0v_1, \dots, v_{r-1}v_r\}$. Now if $v_sv_t \in E(F_d(\Phi))$ with $v_s < v_t$ then by (24) either v_sv_t or $v_t(v_s + j)$ is in $E(\Phi)$. Therefore by the fact that $Z(\Phi)$ is a component of Φ and by (17), either v_sv_t or $v_t(v_s + j)$ is in $E(Z(\Phi))$. Thus $v_sv_t \in E(F_d(Z(\Phi)))$. Applying (17) again gives the conclusion.

Let l be the number of diagonals of $F_d(\Phi)$. Suppose $r \geq 3$. In this case an edge $xy \in E(\Phi)$ is a diagonal of Φ if and only if $f_d(xy)$ is a diagonal of $F_d(\Phi)$. By (26), for each diagonal ab of $F_d(\Phi)$, the preimage $f_d^{-1}[ab]$ consists of d diagonals of Φ . Furthermore, if $ab \neq a'b'$ then $f_d^{-1}[ab]$ and $f_d^{-1}[a'b']$ are disjoint. Thus $k = dl$. Now suppose $r = 2$. If ab is a diagonal of $F_d(\Phi)$ with $ab \neq v_0v_1$ then $f_d^{-1}[ab]$ consists of d diagonals of Φ . Note that either v_0v_1 or $v_1(v_0 + j)$ is a diagonal of Φ , since otherwise $k = 0$. If both v_0v_1 and $v_1(v_0 + j)$ are diagonals then $f_d^{-1}[v_0v_1]$ consists of $2d$ diagonals of Φ . If only one of v_0v_1 and $v_1(v_0 + j)$ is a diagonal of Φ then v_0v_1 is not a diagonal of $F_d(\Phi)$ and $f_d^{-1}[v_0v_1]$ consists of d diagonals of Φ . Thus in either case for $r = 2$, $k = dl + d$ and the result follows. \square

Lemma 22 shows that $F_d^* : G_{\neq d}(n, k; \rho^j) \rightarrow G^*(j)$. We next define an “unfurling” map U_d .

Definition 23. Let $d, j \geq 2$. Define $U_d : G^*(j) \rightarrow G(n)$ as follows. Let $(\Theta, \beta) \in G^r(j)$, and denote the vertices of β by $v_0 < \dots < v_{r-1}$. Define $U_d(\Theta, \beta)$ by $V(U_d(\Theta, \beta)) = \{0, \dots, n-1\}$ and $E(U_d(\Theta, \beta)) = \bigcup_{ab \in E(\Phi)} f_d^{-1}[ab]$, where f_d^{-1} is given by (26).

Lemma 24. Let $d, j \geq 2$. If $(\Theta, \beta) \in G^*(j)$ then $F_d^*(U_d(\Theta, \beta)) = (\Theta, \beta)$.

Proof. Clearly $V(F_d(U_d(\Theta, \beta))) = V(\Theta)$ and

$$E(F_d(U_d(\Theta, \beta))) = f_d[E(U_d(\Theta, \beta))] = f_d \left[\bigcup_{ab \in E(\Theta)} f_d^{-1}[ab] \right] = E(\Theta),$$

so $F_d(U_d(\Theta, \beta)) = \Theta$. Suppose $(\Theta, \beta) \in G^r(j)$; denote the vertices of β by $v_r = v_0 < \dots < v_{r-1}$, and let γ be the graph given by (25). By definition γ is a subgraph of $U_d(\Theta, \beta)$. As in the the proof of Lemma 22, the condition (17) can be used to show that in fact γ is a component of $U_d(\Theta, \beta)$. Finally since the vertices of γ include the regular d -gon with vertices $v_0, v_0 + j, \dots, v_0 + (d-1)j$, their convex hull contains the origin, so $\gamma = Z(U_d(\Theta, \beta))$. Thus $F_d(Z(U_d(\Theta, \beta))) = F_d(\gamma) = \beta$, completing the proof. \square

Lemma 25. Let $d, j \geq 2$. If $\Phi \in G_{\neq d}(n; \rho^j)$ then $U_d(F_d^*(\Phi)) = \Phi$.

Proof. Let $\Phi \in G_{rd}(n; \rho^j)$. It is easily seen that $V(U_d(F_d^*(\Phi))) = V(\Phi)$. As above, the center $Z(\Phi)$ has the form (25). Therefore $V(F_d(Z(\Phi))) = \{v_0, \dots, v_{r-1}\}$, and by Definition 23,

$$E(U_d(F_d^*(\Phi))) = E(U_d(F_d(\Phi), F_d(Z(\Phi)))) = \bigcup_{ab \in E(F_d(\Phi))} f_d^{-1}[ab] = E(\Phi).$$

\square

Theorem 26. Let $1 \leq k \leq n-3$, let $r \geq 2$ and let $2 \leq d \leq n/3$ with $d|n$. Then there exists a bijection:

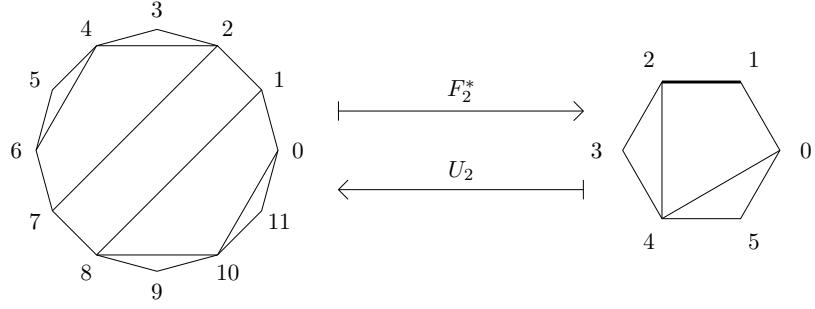
$$G_{rd}(n, k; \rho^j) \longleftrightarrow G^r(j, k/d - \delta_{r2}).$$

Proof. By (23) and Lemmas 22, 24 and 25, the bijection is given in one direction by F_d^* and in the other direction by U_d . \square

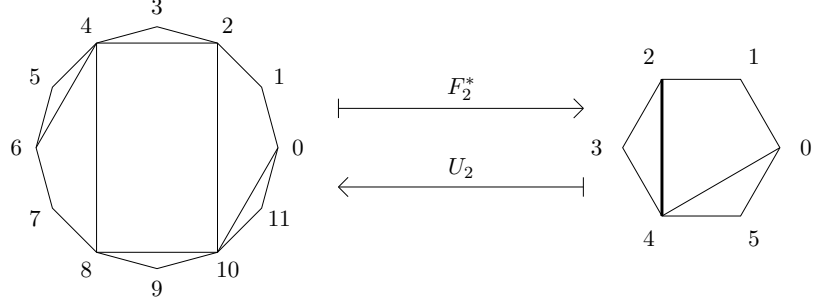
Lemma 12 and Theorem 26 imply that for $d \leq n/3$,

$$\begin{aligned} |G_{\neq d}(n, k; \rho^{n/d})| &= |G^2(n/d, k/d - 1)| + \sum_{r \geq 3} |G^r(n/d, k/d)| \\ &= \frac{n+k-d}{d} A_{k/d-1}^{n/d} + \sum_{\substack{r \geq 3; n_1 + \dots + n_r = n/d \\ k_1 + \dots + k_r + |\{i | n_i \geq 2\}| = k/d}} \frac{n}{r} \prod_{i=1}^r A_{k_i}^{n_i+1}. \end{aligned} \tag{27}$$

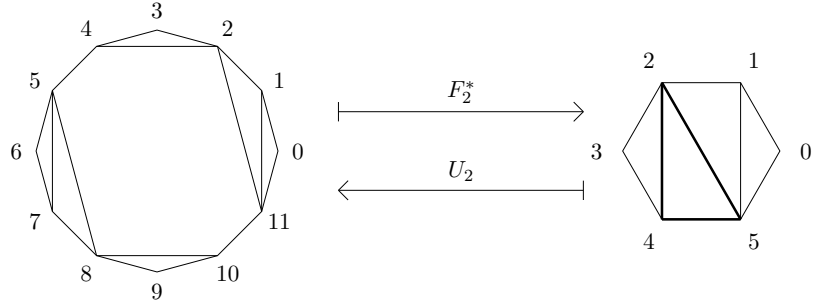
Note that if $d > n/3$ then $|G_{\neq d}(n, k; \rho^{n/d})| = 0$.



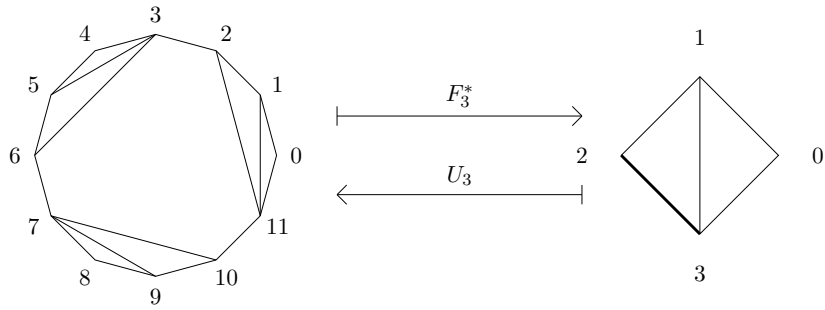
(a) $G_4(12, 6; \rho^6) \leftrightarrow G^2(6, 2)$



(b) $G_4(12, 6; \rho^6) \leftrightarrow G^2(6, 2)$



(c) $G_6(12, 6; \rho^6) \leftrightarrow G^3(6, 3)$



(d) $G_6(12, 6; \rho^4) \leftrightarrow G^2(4, 1)$

Figure 3: Examples showing the bijections of Theorem 26.

5.1. Proof of Theorems 6 and 7

The proof of Theorems 6 and 7 now follows by substituting into equations (11) and (12) the expressions obtained in (13)–(16) for the number of axially symmetric dissections, and the values obtained in (22) and (27) for rotationally symmetric dissections.

6. Interesting special cases

The enumeration formulas can be specialized to certain classes of dissections, namely for specific values of $n - k$ and for specific values of k . The next lemma is equivalent to Catalan's k -fold convolution formula [5, 10].

Lemma 27. *For any $n, m \geq 0$,*

$$\sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} C_{i_1} \dots C_{i_m} = \begin{cases} \frac{m(n+1)(n+2) \dots (n + \frac{m}{2} - 1)}{2(n + \frac{m}{2} + 2)(n + \frac{m}{2} + 3) \dots (n+m)} C_{n+m/2}, & \text{if } m \text{ is even} \\ \frac{m(n+1)(n+2) \dots (n + \frac{m-1}{2})}{(n + \frac{m+3}{2})(n + \frac{m+3}{2} + 1) \dots (n+m)} C_{n+(m-1)/2}, & \text{if } m \text{ is odd.} \end{cases}$$

Lemma 28. 1. *For any $n \geq 2$,*

$$A_{n-3}^n + A_{n-2}^n = C_{n-2}. \quad (28)$$

2. *For any $n \geq 2, q \geq 2$,*

$$\sum_{i+j=n} A_{i-1}^{i+1} A_{j+1-q}^{j+1} = A_{n-q}^n. \quad (29)$$

3. *For any $n \geq 3$,*

$$\sum_{i+j=n} A_{i-2}^{i+1} A_{j-2}^{j+1} = C_{n-1} - 2C_{n-2}. \quad (30)$$

4. *For any $n \geq 3$,*

$$\sum_{i+j=n} A_{i-2}^{i+1} A_{j-3}^{j+1} = \frac{(n-3)(n-4)}{2n} C_{n-2}. \quad (31)$$

Proof. Equations (28) and (29) follow from (5), and (30) follows from (28) and from Lemma 27. To prove (31), we show that

$$(n-4)A_{n-4}^n = n \sum_{i+j=n} A_{i-2}^{i+1} A_{j-3}^{j+1}. \quad (32)$$

The result will then follow since $A_{n-4}^n = \frac{n-3}{2} C_{n-2}$. Now the left hand side of (32) is the number of almost-triangulations marked by a diagonal (i.e., $(\Phi, \beta) \in G^2(n, n-4)$ where $V(\beta)$ is not of the form $\{v, [v+1]_n\}$). These can also be enumerated as follows. Choose one vertex v out of the n vertices, then choose $2 \leq i \leq n-3$ and $j = n-i$. Mark the diagonal $v[v+i]_n$, and choose a triangulation of the resulting $(i+1)$ -gon and an almost-triangulation of the resulting $(j+1)$ -gon. \square

The details of the proof of Theorem 3 for n even are given below. Most of the details of the other cases are omitted. Note that if

$$\sum_{\substack{n_1+\dots+n_r=n/d \\ k_1+\dots+k_r+|\{i:n_i\geq 2\}|=k/d}} A_{k_i}^{n_i+1} \neq 0$$

for some $d \geq 2$, $r \geq 3$, then it follows from (21) that $6 \leq rd \leq n - k$. Thus these terms vanish in the cases $k = n - 3$, $n - 4$ or $n - 5$.

Applying (6) and (7) when $k = n - 3$ (i.e., for triangulations), recovers the result of Moon and Moser and the result of Brown. (We omit the details here as the even dihedral case of Theorem 3, for which the details are provided, is a similar calculation).

Theorem 29. 1. [13] *Let $n \geq 3$. The number of triangulations of an n -gon (equivalently, the number of vertices of the $(n - 3)$ -dimensional associahedron) modulo the dihedral action is*

$$|G(n, n - 3)/D_{2n}| = \begin{cases} \frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{3}{4}C_{n/2-1}, & \text{if } n \text{ is even} \\ \frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{1}{2}C_{(n-3)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

2. [4] *Let $n \geq 3$. The number of triangulations of an n -gon (equivalently, the number of vertices of the $(n - 3)$ -dimensional associahedron) modulo the cyclic action is*

$$|G(n, n - 3)/Z_n| = \frac{1}{n}C_{n-2} + \frac{1}{2}C_{n/2-1} + \frac{2}{3}C_{n/3-1}.$$

Setting $k = n - 4$ in (6) recovers the following result of the authors.

Theorem 30. [3] *Let $n \geq 4$, and let $g^{(e)}(n)$ be the number of almost-triangulations of an n -gon (equivalently, edges of the $(n - 3)$ -dimensional associahedron) modulo the dihedral action. Then*

$$g^{(e)}(n) = \begin{cases} (\frac{1}{4} - \frac{3}{4n})C_{n-2} + \frac{3}{8}C_{n/2-1} + (1 - \frac{3}{n})C_{n/2-2} + \frac{1}{4}C_{n/4-1}, & \text{if } n \text{ is even} \\ (\frac{1}{4} - \frac{3}{4n})C_{n-2} + \frac{1}{4}C_{(n-3)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

Setting $k = n - 4$ in (7) gives the result of Theorem 2. The following proposition gives the details needed to simplify (6), thus completing the proof of the even dihedral case of Theorem 3. The other cases are easier.

Proposition 31. *Let $n \geq 6$ and $k = n - 5$. If n is even then*

$$\frac{1}{2n}A_k^n = \frac{(n - 3)^2(n - 4)}{8n(2n - 5)}C_{n-2}, \quad (33)$$

$$\frac{1}{2}A_{(k-1)/2}^{n/2+1} = \frac{n - 4}{8}C_{n/2-1}, \quad (34)$$

$$\frac{1}{4}A_{k/2}^{n/2+1} = 0, \quad (35)$$

$$\sum_{3 \leq d|n} \frac{\varphi(d)}{2d} A_{k/d-1}^{n/d+1} = \frac{2}{5} C_{n/5-1}, \quad (36)$$

$$\sum_{2 \leq d \leq n/3} \frac{\varphi(d)(n+k-d)}{2dn} A_{k/d-1}^{n/d} = 0, \quad (37)$$

$$\sum_{\substack{2 \leq d|n; \ r \geq 3; \ n_1 + \dots + n_r = n/d \\ k_1 + \dots + k_r + |\{i: n_i \geq 2\}| = k/d}} \frac{\varphi(d)}{2r} \prod_{i=1}^r A_{k_i}^{n_i+1} = 0, \quad (38)$$

$$\frac{1}{4} \sum_{\substack{1 \leq t \leq k \\ n_0 + \dots + n_t = n/2 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}) = \frac{n^2 - 2n - 12}{16(n-3)} C_{n/2-1}, \quad (39)$$

and

$$\frac{1}{4} \sum_{\substack{0 \leq t \leq k \\ n_0 + \dots + n_t = n/2-1 \\ k_0 + \dots + k_t = (k-t)/2}} \prod_{s=0}^t (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}) = \frac{n}{16(n-3)} C_{n/2-1}. \quad (40)$$

Proof. Equation (33) follows from (5), and (34) and (35) are immediate. In (36), the only nonzero summand corresponds to $d = 5$. To prove (37), note that if d divides n and k then $d = 5$, but then $A_{k/d-1}^{n/d} = 0$ since $n/d \geq 3$. Equation (38) follows from the remark before Theorem 29. For (39), note that if $\prod_{s=0}^p (A_{k_s-1}^{n_s+1} + A_{k_s}^{n_s+1}) \neq 0$, then $k_s \leq n_s - 1$ for all s . Therefore in this case

$$(k-t)/2 = \sum_{s=0}^t k_s \leq \sum_{s=0}^t (n_s - 1) = n/2 - t,$$

i.e., $t \leq n - k - 2$, with equality if and only if $k_s = n_s - 1$ for all s . Therefore, using the notation of Section 3, the nonzero summands in (39) correspond to $|G(n, n-5; \tau; 1)|$ and $|G(n, n-5; \tau; 3)|$. Now

$$\begin{aligned} |G(n, n-5; \tau; 1)| &= \sum_{\substack{n_0+n_1=n/2 \\ k_0+k_1=n/2-3}} (A_{k_0-1}^{n_0+1} + A_{k_0}^{n_0+1}) (A_{k_1-1}^{n_1+1} + A_{k_1}^{n_1+1}) \\ &= \sum_{\substack{n_0+n_1=n/2 \\ k_0+k_1=n/2-3}} (A_{k_0-1}^{n_0+1} A_{k_1-1}^{n_1+1} + A_{k_0-1}^{n_0+1} A_{k_1}^{n_1+1} + A_{k_0}^{n_0+1} A_{k_1-1}^{n_1+1} + A_{k_0}^{n_0+1} A_{k_1}^{n_1+1}). \end{aligned} \quad (41)$$

Any nonzero terms in (41) have $(k_0, k_1) = (n_0 - 1, n_1 - 2)$ or $(k_0, k_1) = (n_0 - 2, n_1 - 1)$. Therefore by Lemma 28 and by symmetry,

$$\begin{aligned} |G(n, n-5; \tau; 1)| &= 2 \sum_{n_0+n_1=n/2} (A_{n_0-2}^{n_0+1} A_{n_1-3}^{n_1+1} + A_{n_0-2}^{n_0+1} A_{n_1-2}^{n_1+1} + A_{n_0-1}^{n_0+1} A_{n_1-3}^{n_1+1} + A_{n_0-1}^{n_0+1} A_{n_1-2}^{n_1+1}) \\ &= 2 \left[\frac{(n-6)(n-8)}{4n} C_{n/2-2} + C_{n/2-1} - 2C_{n/2-2} + \frac{n-6}{4} C_{n/2-2} + C_{n/2-2} \right]. \end{aligned}$$

Next by Lemma 27,

$$\begin{aligned} |G(n, n-5; \tau; 3)| &= \sum_{n_0+n_1+n_2+n_3=n/2} (A_{n_0-2}^{n_0+1} + A_{n_0-1}^{n_0+1}) (A_{n_1-2}^{n_1+1} + A_{n_0-1}^{n_0+1}) (A_{n_2-2}^{n_2+1} + A_{n_0-1}^{n_0+1}) (A_{n_3-2}^{n_3+1} + A_{n_0-1}^{n_0+1}) \\ &= \sum_{n_0+n_1+n_2+n_3=n/2} C_{n_0-1} C_{n_1-1} C_{n_2-1} C_{n_3-1} = \frac{2n-12}{n} C_{n/2-2}. \end{aligned}$$

Equation (39) now follows by simplifying these expressions and using the relation $C_{n/2-2} = \frac{n}{4(n-3)} C_{n/2-1}$. A similar argument proves (40). \square

The proof of Theorem 3 now follows by applying Proposition 31 and Theorems 6 and 7. The proof of Theorems 4 and 5 proceeds along similar lines. For Theorem 4, note that if $|G^r(n/d, k/d - \delta_{r,2})| \neq 0$ then either $r = 2$ and $d = 2$; or $r = 2$ and $d = 3$; or $r = 3$ and $d = 2$ (see Figure 3). The last of these cases can be computed using (20):

$$|G^3(n/2, (n-6)/2 - 3 + 3)| = (n/2 - 2) A_{n/2-3}^{n/2} = (n/2 - 2) C_{n/2-2}.$$

References

- [1] M. Aigner, A Course in Enumeration, Springer 2007.
- [2] W. G. Brown, Enumeration of triangulations of the disk, Proc. London Math. Soc. (3), (1964) 746-768.
- [3] D. Bowman and A. Regev, Counting equivalence classes of vertex pairs modulo the dihedral action on the associahedron, Proc. Amer. Math. Soc. (to appear).
- [4] W. Burnside, Theory of Groups of Finite Order, 2nd ed., Cambridge University Press, Cambridge, 1911. p. 191.
- [5] E. Catalan, Sur les nombres de Segner, Rend. Circ. Mat. Palermo, 1 (1887) 190-201.
- [6] A. Cayley, On the partitions of a polygon, Proc. Lond. Math. Soc. 22 (1890-91) 237-262.
- [7] C. Ceballos, F. Santos, and G. M. Ziegler, Many non-equivalent realizations of the associahedron, (2011) <http://arxiv.org/abs/1109.5544>.

- [8] S. L. Devadoss and R. C. Read, Cellular structures determined by polygons and trees, *Ann. Comb.* 5 (2001) 71-98.
- [9] J. A. De Loera, J. Rambau, and F. Santos, *Triangulations: Structures for Algorithms and Applications*, Springer-Verlag, Berlin, 2010.
- [10] P. J. Larcombe and D. R. French, The Catalan number k -fold self-convolution identity: the original formulation, *J. Combin. Math. Combin. Comput.*, 46 (2003) 191–204.
- [11] C. Lee, The associahedron and triangulations of the n -gon, *European J. Combin.* 10 (1989), no. 6, 551-560.
- [12] P. Lisonek, Closed forms for the number of polygon dissections. *J. Symbolic Comput.* 20 (1995), 595-601.
- [13] J. W. Moon and L. Moser, Triangular dissections of n -gons, *Canad. Math. Bull.*, 6 (1963), 175-178.
- [14] J. H. Przytycki and A. Sikora, Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers, *J. Comb. Theory A*, 92 (1) (2000), 68–76.
- [15] R. C. Read, On general dissections of a polygon, *Aequat. Math.* 18 (1978), 370-388.
- [16] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>.
- [17] H. Torkildsen, Counting cluster-tilted algebras of type A_n , *Int. Electron. J. Algebra*, 4(2008), 149-158.